

# On the Quantum Creation of Matter in the Expanding Universe

Natalia Gorobey and Alexander Lukyanenko<sup>1,\*</sup>

<sup>1</sup>*Department of Experimental Physics, St. Petersburg State Polytechnical University,  
Polytekhnicheskaya 29, 195251, St. Petersburg, Russia*

Quantum Action Principle which has been used as a ground for a probabilistic interpretation of one-particle relativistic quantum mechanics [4] is applied to quantum cosmology. The quantum creation of matter in a minisuperspace model with one homogeneous scalar field is considered. The initial state of the universe is defined as a stationary ground state of the Hamiltonian with the Euclidean signature in which the mean value of the universe radius is equal to the Plank length and the number of the scalar field quanta is equal zero. We interpret the change of the signature as the universe "birth". From this moment of time the dynamics of the scale factor is considered as classical. The real phase of the amplitude of the creation process is taken as a quantum action. The balance between matter and gravitation energies in the creation process is fulfilled by the condition of the stationarity of the quantum action with respect to the internal time of the universe.

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## I. INTRODUCTION

The idea of the quantum creation of matter in the expanding universe has been developed in 1970's [1]. The creation was considered as a parametric excitation of quantum matter fields on the background of the classical Friedman expanding universe. But the effectiveness of this mechanism of excitation is not high [2], at least for the ordinary matter. The rate of the matter creation is maximum near the singularity, but the latter is a problem in classical cosmology. This approach by itself is considered as an approach to precise quantum cosmology (QC) with the quantized space-time metrics. In the precise theory constraints (quantum constraints) play the central role and determine the dynamics of the universe [5]. Specifically, the Hamiltonian constraint ensures the balance between matter and gravitational field energies in the universe. This balance was not taken into account in the approach mentioned above [1]. To take it into account we don't need precise QC. The goal may be achieved by means of a quantum action principle (QAP) in which an analog of the classical constraints is obtained at the quantum level as a set of conditions of stationarity of a quantum action [3]. From the first time QAP was used for a probabilistic interpretation of one-particle relativistic quantum mechanics in [4]. A quantum action may be built in a semi-classical form with a classical gravitational part and a quantum matter addition. This semi-classical quantum action will define the dynamics of the classical geometry of the universe with account of a quantum matter back-reaction, and it has to be stationary with respect to Lagrangian multipliers  $N_\mu$ , which are coefficients in front of the classical constraints in the classical action of General Relativity [5]. However, the situation near the singularity must be considered in the framework of precise QC. The present work is devoted

to application of QAP to the simplest minisuperspace model of the universe - the Friedman universe with one homogeneous scalar field. We make two improvements in the original approach to the quantum creation of matter. The first one - we consider the scalar field as the only matter contents of the universe, and take into account a back-reaction of its quantum dynamics on the classical dynamics of the scale factor. The second - the energy balance between the matter and geometry is ensured by a condition of stationarity of a quantum action with respect to an internal proper time of the universe. In order to solve the problem of the initial state of the universe near the singularity, in the next section we begin with formulation of full quantum theory of the Friedman universe. Then we come to a semi-classical description of the dynamics of the Friedman universe in analogy with the dynamics of the Minkowsky time parameter  $x^0$  of a relativistic particle, or a bosonic string [3].

## II. MINISUPERSPACE MODEL OF THE UNIVERSE

The classical dynamics of the Friedman universe with the space-time interval

$$ds^2 = N^2(t) dt^2 - a^2(t) [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2)] \quad (1)$$

where  $N(t)$  is the lapse function, and  $a(t)$  is the spatial scale factor, with one scalar field  $\phi$  is described by the classical action (velocity of light equals unity) [5]:

$$I = \int \left[ \frac{1}{2g} \left( aN - a \frac{\dot{a}^2}{N} \right) + \frac{1}{2} 2\pi^2 a^3 \left( \frac{\dot{\phi}^2}{N} - m^2 N \phi^2 \right) \right] dt, \quad (2)$$

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\*Electronic address: alex.lukyan@rambler.ru

where  $g \equiv 3\pi/2G$ ,  $G$  is the Newton gravitational constant. The canonical form of the action (2) is

$$I = \int_0^C dc \left( p_a \dot{a} + p_\phi \dot{\phi} - H \right), \quad (3)$$

where

$$H \equiv -\frac{1}{2} \left( \frac{gp_a^2}{a} + \frac{a}{g} \right) + \frac{1}{2} \left( \frac{p_\phi^2}{2\pi^2 a^3} + 2\pi^2 a^3 m^2 \phi^2 \right) \quad (4)$$

is the Hamiltonian constraint, which in fact regulates the balance of matter and gravitational field energies at the classical level. It must be equal zero as a condition of stationarity with respect to the Lagrangian multiplier  $N$  [5]. In equation (4) we introduced a time parameter  $c$ ,  $c \in [0, C]$ , which is related to the ordinary time as follows,  $dc = Ndt$  [6]. In this case the upper limit is a free dynamical variable.

Let us turn to quantum theory. Introducing the operators of momenta,

$$\hat{p}_a \equiv \frac{\hbar}{i} \frac{\partial}{\partial a}, \hat{p}_\phi \equiv \frac{\hbar}{i} \frac{\partial}{\partial \phi}, \quad (5)$$

we define the Hamiltonian operator as follows:

$$\hat{H} = -\hat{H}_a + \hat{H}_\phi \quad (6)$$

$$\hat{H}_a \equiv \frac{1}{2} \left( -\hbar^2 g \frac{1}{a^q} \frac{\partial}{\partial a} a^{q-1} \frac{\partial}{\partial a} + \frac{a}{g} \right) \quad (7)$$

$$\hat{H}_\phi \equiv \frac{1}{2} \left( -\frac{\hbar^2}{2\pi^2 a^3} \frac{\partial^2}{\partial \phi^2} + 2\pi^2 a^3 m^2 \phi^2 \right) \quad (8)$$

Here we chose a certain ordering of non-commuting operator multipliers in (7), assuming the integration measure on the minisuperspace  $(a, \phi)$  to be

$$a^q da d\phi. \quad (9)$$

We interpret the scale factor  $a$ ,  $a \in [0, \infty)$  as a "radial" coordinate in the superspace, and take for definiteness  $q = 3$ . Quantum dynamics of the universe in our theory is described by a wave function  $\psi(c, a, \phi)$ , which is a solution of the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial c} = \hat{H} \psi. \quad (10)$$

with corresponding initial data. Let us mention that  $c \in [0, C]$ , and  $C$  is up to now arbitrary.

The dynamical parameter  $C$  is not observable and must be excluded in the framework of QAP [3]. The role of observable time in our theory will play the scale factor  $a$ . But one can use the description of the quantum dynamics in terms of the formal time parameter  $c$  in the Schrödinger equation (10) for regularization of the

theory near the singularity  $a = 0$ . In order to make this regularization, let us slightly "improve" the theory by introducing a signature function in the Hamiltonian (6):

$$\hat{H}(c) = f(c) \hat{H}_a + \hat{H}_\phi, \quad (11)$$

where, for example,

$$f(c) = 2 \exp \left( -\frac{c}{c_{pl}} \right) - 1. \quad (12)$$

Near the singularity,  $c \rightarrow 0$ , the "improved" Hamiltonian (11) is a positive definite operator. It will be used in the next section for determining the initial quantum state of the universe at the moment  $c = 0$ . After the Planckian epoch,  $c \gg c_{pl}$ , the Hamiltonian (11) restores its Lorentzian signature.

### III. INITIAL QUANTUM STATE OF THE UNIVERSE

Let the initial state of the universe  $\psi_0(a, \phi)$  be an eigenfunction of the Hamiltonian (11) taken at the moment  $c = 0$ :

$$\hat{H}(0) \psi_0 = E \psi_0 \quad (13)$$

which corresponds to its minimal eigenvalue  $E$ . In analogy with ordinary quantum theory of the hydrogen atom, we expect that the ground state  $\psi_0(a, \phi)$  of the universe is non-singular. In order to find the ground state one can use the ordinary variational principle for the functional

$$F[\psi] \equiv \frac{(\psi, \hat{H}(0) \psi)}{(\psi, \psi)}. \quad (14)$$

An approximation to the exact ground state is given by the following two-parametric Gauss function:

$$\tilde{\psi}_0(a, \phi) = \exp \left( -\frac{\alpha a^2}{2} - \frac{\beta \phi^2}{2} \right), \quad (15)$$

for which the functional (14) equals

$$F = \frac{3}{8g} \sqrt{\frac{\pi}{\alpha}} (\hbar^2 g^2 \alpha^2 + 1) + \frac{\hbar^2}{8\pi\sqrt{\pi}} \alpha^{3/2} \beta + \frac{15}{16} \frac{\pi^{5/2} m^2}{\alpha^{3/2} \beta}. \quad (16)$$

Its minimum value equals

$$F_m = \frac{3^{1/4}}{2} \sqrt{\pi} \sqrt{\frac{\hbar}{g}} + \frac{\sqrt{15\pi}}{4\sqrt{2}} \hbar m, \quad (17)$$

when

$$\alpha = \frac{1}{\sqrt{3}\hbar g}, \beta = \sqrt{\frac{15}{2}} 3^{3/4} \pi^2 m g^{3/2} \sqrt{\hbar} \quad (18)$$

The main result of our consideration is the estimation of mean value of the scale factor in the initial ground state of the universe: it has a non-zero value of the order of the Plank length,

$$\langle a \rangle \equiv \frac{(\psi_0, a\psi_0)}{(\psi_0, \psi_0)} = \frac{3^{5/4}}{4} \sqrt{\pi} \sqrt{\hbar g} \simeq l_{pl}. \quad (19)$$

The energy of the scalar field in the ground state (the second term in (17)) is close to the vacuum energy  $(1/2) \hbar m$ . Therefore, in analogy with the ordinary quantum theory of the hydrogen atom the initial ground state of the universe is non-singular. The universe will remain in the ground state up to the moment  $c = c_{pl}$ , when the signature of the Hamiltonian (11) will be changed. This moment may be interpreted as "birth" of the universe. Now one can go to the semi-classical description of the universe dynamics.

#### IV. CREATION OF MATTER IN THE EXPANDING UNIVERSE

Let us turn to the second stage of the universe dynamics after its "birth". Let us shift the initial moment of the proper time to the moment of "birth", and consider the universe dynamics on the interval  $c \in [0, C]$  in the semi-classical approximation proposed in [3]. The semi-classical approximation in the framework of QAP is achieved by means of an "improvement" of the original classical action before quantization. We modify the first part (7) of the Hamiltonian, introducing additional complex variables  $\lambda = \lambda_1 + i\lambda_2$  and  $d = d_1 + id_2$  for description of the scale factor dynamics, as follows:

$$\hat{H}'_a = -\frac{1}{2g} |d|^2 + \lambda(d - gp_a - ia) + \bar{\lambda}(\bar{d} - gp_a + ia). \quad (20)$$

It is obvious that at the classical level the modified Hamiltonian is equivalent to the original one, if we consider  $\lambda, d$  as independent dynamical variables with canonical momenta equal zero. But now the Hamiltonian becomes linear with respect to  $p_a$ . The idea [3] is to quantize the theory with the arbitrary variables  $\lambda(c), d(c)$  at the condition that they are fixed at the quantum level in the framework of QAP. The corresponding modified Schrödinger equation may be written in a form:

$$\begin{aligned} & \left( i\hbar \frac{\partial}{\partial c} - 2i\hbar g \lambda_1 \frac{\partial}{\partial a} \right) \psi \\ &= \left[ -\frac{1}{2ga} (d_1^2 + d_2^2) + (\lambda_1 d_1 - \lambda_2 d_2) + 2a\lambda_2 + \hat{H}_\phi \right] \psi. \end{aligned} \quad (21)$$

Now, one can consider the scale factor as a function of time  $a(c)$  and the expression in the round brackets in the left hand side of (21) as a full derivative of a wave function  $\psi(c, a(c), \phi)$  with respect to  $c$ , if we take  $-2g\lambda_1 \equiv \dot{a}$ . The function  $a(c)$  must be defined by means of QAP, as well.

The quantum action in QAP is defined as the real phase of the transition amplitude for a given quantum transition [3]. Let us consider the transition of the quantum oscillator  $\phi$  from the (normalized) vacuum state  $|0\rangle$ ,

$$|0\rangle = \pi^{-1/4} \exp\left(-\frac{1}{2} 2\pi^2 a_0^3 \frac{m}{\hbar} \phi^2\right), \quad (22)$$

at the moment  $c = 0$  with  $a_0 = l_{pl}$  to an excited (normalized) state  $|n\rangle$  at the moment  $c = C$ ,

$$|n\rangle = \frac{\pi^{-1/4}}{\sqrt{2^n n!}} H_n \left( \sqrt{2\pi^2 a_1^3 \frac{m}{\hbar}} \phi \right) \exp\left(-\frac{1}{2} 2\pi^2 a_1^3 \frac{m}{\hbar} \phi^2\right), \quad (23)$$

where  $a_1 \equiv a(C)$  is a final value of the scale factor. It is this quantity that will play the role of an observable time parameter in the expanding universe. The corresponding transition amplitude

$$A_{n0} \equiv \langle n | \hat{U}_C | 0 \rangle, \quad (24)$$

where  $\hat{U}_C$ , is the evolution operator of the modified Schrödinger equation (21) in the interval  $[0, C]$ , can be written in a form:

$$A_{n0} = \exp\left(\frac{i}{\hbar} \Lambda_{Ca}\right) \langle n | \hat{U}_{C\phi} | 0 \rangle, \quad (25)$$

where  $\hat{U}_{C\phi}$  is the evolution operator of the Schrödinger equation for only scalar field part on a classical homogeneous space-time background with arbitrary function  $a(c)$ :

$$i\hbar \frac{\partial \psi}{\partial c} = \hat{H}_\phi \psi, \quad (26)$$

and

$$\Lambda_{Ca} \equiv \int_0^C dc \left[ \frac{1}{2ga} (d_1^2 + d_2^2) + \left( \frac{\dot{a}}{2g} d_1 + \lambda_2 d_2 \right) - 2a\lambda_2 \right]. \quad (27)$$

At this stage we can find the stationary value of the phase (27) as a function of additional variables  $d_{1,2}, \lambda_2$ . A resulting quantity is equal to the gravitational part of the original classical action (2).

It is the equation (26), that describes the creation of matter in the expanding universe (minisuperspace model) in the approach mentioned above [1], at the condition that the dynamics of  $a(c)$  is classical. The parametric excitation of the scalar field  $\phi$  arises due to the dependence of  $\hat{H}_\phi$  on  $a(c)$  in accordance with (8). If we only take into account this mechanism of excitation, we will lose the balance between matter and gravitation field energies, that is regulated by the equation (4) in classical theory. We shall restore this balance by taking into account the dynamics of the scale factor  $a(c)$  in the framework of QAP, and the important additional condition of stationarity of a quantum action with respect to

$C$ . Therefore, in our approach we have overcome only half of the way.

Let us turn to the formulation of QAP. We define the full quantum action as a sum,

$$\Lambda = \Lambda_{Ca} + \Lambda_{C\phi}. \quad (28)$$

Here  $\Lambda_{Ca}$  is the classical action for the scale factor  $a(c)$ ,  $\Lambda_{C\phi}$  is the real phase of the transition amplitude  $\langle n | \hat{U}_{C\phi} | 0 \rangle$  written in the exponential form:

$$\langle n | \hat{U}_{C\phi} | 0 \rangle = R_{C\phi} \exp \left( \frac{i}{\hbar} \Lambda_{C\phi} \right). \quad (29)$$

In analogy with classical action principle, we formulate QAP as a set of the stationarity conditions of the quantum action:

$$\frac{\delta \Lambda}{\delta a(c)} = 0, \frac{\partial \Lambda}{\partial C} = 0. \quad (30)$$

The first equation in (30) defines the (semi-classical) dynamics of the scale factor  $a(c)$  with fixed boundary values  $a(0) = a_0, a(C) = a_1$ . This equation of motion takes into account the back-reaction of quantum dynamics of the scalar field  $\phi$ . The second equation in (30) fixes the proper time  $C$ . It is this condition, that restores the balance of energies in the creation of matter process. The balance is restored by tuning of the proper time  $C$  to given parameters of the final state of the universe, i.e. the quantum number  $n$ , and the spatial size of the universe  $a_1$ . Let us stress that the semi-classical history of the scale factor and the length of the proper time interval are defined for a given quantum transition  $|0\rangle \rightarrow |n\rangle$ . We must substitute these quantities into the amplitude (29).

Calculation of probabilities in QAP needs a special consideration. The quantum evolution of matter described by the Schrödinger equation (26) is unitary. But additional conditions (30) and a posteriori manipulations with amplitude (29) destroy the unitarity. We

must renormalize all amplitudes (29) ( $n = 0, 1, 2, \dots$ ) after taking into account the back-reaction of the creation of matter process on the dynamics of the scale factor and the energy balance. The properly normalized transition amplitude of the process  $|0\rangle \rightarrow |n\rangle$  is

$$K_n(a_1) = \frac{1}{\sqrt{Z}} \langle n | \hat{U}_{C\phi} | 0 \rangle, Z \equiv \sum_{n=0}^{\infty} \left| \langle n | \hat{U}_{C\phi} | 0 \rangle \right|^2. \quad (31)$$

This is the probability density to detect  $n$  quanta of the homogeneous scalar field  $\phi$  at the moment when the radius of the universe will be equal  $a_1$ . We expect that the probability will be maximum for states with energy equal to the gravitational energy corresponding to a macroscopic value of the scale factor  $a_1$ .

## V. CONCLUSIONS

The balance between matter and gravitational energies in the universe, which is guaranteed by the Hamiltonian constraint equation in classical General Relativity is realized in the present work at the quantum level as a condition of the stationarity of a quantum action with respect to an internal time of the universe. The latter is not observable quantity in a quantum universe. The role of observable time in the universe plays its spatial scale factor  $a$ . We expect that the properly normalized transition amplitude (31), which takes into account the energy balance, will give us a sufficient rate for the quantum creation of matter in the expanding universe.

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- [1] A.A.Grib, S.G.Mamaev, and V.M.Mostepanenko *Quantum Effects in Intense External Fields* (Soviet edition of Moskow Atomizdat 1980).
  - [2] S.Wainberg *The First Three Minutes. A Modern View on the Origin of the Universe* (ISBN 0-465-02437-8, 1977, updated with new afterword in 1993).
  - [3] Natalya Gorobey, and Alexander Lukyanenko, arXiv: 1103.0132v1[quant-ph].
  - [4] Natalia Gorobey, Alexander Lukyanenko, and Inna Lukyanenko, arXiv: 1010.3824v1[quant-ph].
  - [5] C.W.Misner, K.S.Thorne, and J.A.Wheeler, *Gravitation*, (W.H.Freeman and Company, San-Francisco 1073).
  - [6] V.Fock, *Works on Quantum Field Theory* (Soviet edition of the Leningrad State Univ. 1957).